Non Gaussianity of General Multiple-Field Inflationary Models

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Abstract

Using the "δN-formalism", We obtain the expression of the non-Gaussianity of multiple-field inflationary models with the nontrivial field-space metric. Further, we rewritten the result by using the slow-rolling approximation.

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Key words: non Gaussian, non linear parameter, nontrivial metric

1 Introduction

In modern cosmology, the inflation paradigm plays an important role. The simplest classes of inflation models predict Gaussian-distributed perturbations and a nearly scale-invariant spectrum of the primordial density perturbations [1]. This is in good agreement with cosmological observations [2]. Despite the appealing simplicity behind the central idea of inflation, it has

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proved difficult to discriminate between the large number of different models that have been developed to date [4]. However, it is believed that the deviation away from the Gaussian statistics represents a potential powerful discriminant between the competing inflationary models. At the same time, it is necessary to extend the theoretical framework beyond the leading-order effects of scale-invariant, Gaussian fluctuations, in order to understand the early universe further. So, recently, the non-Gaussianity has attracted considerable interest. (See [5] for a review.) In [6], Maldacena gave a general analysis of non-Gaussian perturbations in single field inflationary models. His result is that the bispectrum of the curvature perturbation for squeezed triangles ($k_1 \ll k_2, k_3$) is proportional to the tilt of the primordial power spectrum, and hence is small. Then it seems that only multiple-field inflationary models is like to generate significant non-Gaussian perturbations [7, 8, 9, 10, 11, 12]. In [11], Lyth and Rodriguez have shown that the non-Gaussianity of the curvature perturbation in multiple field models can be simply expressed in the so-called "δN-formalism" [13]. There the "separate universe" approach [14] has been used to define the curvature perturbation. It should be noted that this approach is valid only for perturbations on super-Hubble scales. But Lyth and Rodriguez only express f_{NL} on a field space with the trivial metric, where f_{NL} is the non-linear parameter. In [8], the authors have given the expression of f_{NL} involving the metric of the field space, G_{IJ}, explicitly. But they restricted their attentions only on the metric that *can be brought to the field-independent form* $G_{IJ} = \delta_{IJ}$ *by an appropriate choice of parametrization.*

In this paper, first, the result in [11] is generalized to the case with a generic field-space metric. It is found that the generalized expression is similar to the expression obtained in [8] for a trivial field-space metric. Then this expression is rewritten in terms of slow-rolling parameters.

2 The background and the curvature perturbation

The starting point is the effective action of the simple coupling system of Einstein gravity and scalar fields with an arbitrary inflation potential $V(\varphi^I)$

$$S = \int \sqrt{-g} d^4x [\frac{M_p^2}{2} R - \frac{1}{2} G_{IJ} \partial_\mu \varphi^I \partial^\mu \varphi^J - V(\varphi^I)], \tag{1}$$

where $G_{IJ} \equiv G_{IJ}(\varphi^K)$ represents the metric on the manifold parameterized by the scalar field values, the 'target space' metric, and $8\pi G = M_p^{-2}$ represents the reduced Planck mass. Units are chosen such that $c = \hbar = 1$. For the background model, the Friedmann-Robertson-Walker metric is used,

$$ds^2 = -dt^2 + a^2(t) \delta_{ij} dx^i dx^j. \tag{2}$$

Take the background scalar fields as $\varphi^I(t)$. Then background equations of the scalar fields are

$$\ddot{\varphi}^I + 3H\dot{\varphi}^I + \Gamma^I_{JK} \dot{\varphi}^J \dot{\varphi}^K + G^{IJ} V_{,I} = 0, \tag{3}$$

where $\Gamma^I_{JK} = \frac{1}{2} G^{IL}(G_{JL,K} + G_{LK,J} - G_{JK,L})$, are the target space Christoffel symbols. $H = \dot{a}/a$ is the Hubble parameter, $\dot{\varphi}^I = d\varphi^I/dt$, $\ddot{\varphi}^I = \frac{d^2\varphi^I}{dt^2}$ and $V_{,I} = \frac{\partial V}{\partial \varphi^I}$, $G_{IJ,K} = \frac{\partial G_{IJ}}{\partial \varphi^K}$. Basing on the Einstein equation, we get

$$\frac{\ddot{a}}{a} = -\frac{1}{3M_p^2}(G_{IJ} \dot{\varphi}^I \dot{\varphi}^J - V). \tag{4}$$

Together with the Friedmann equation

$$H^2 = \frac{1}{3M_p^2}(\frac{1}{2} G_{IJ} \dot{\varphi}^I \dot{\varphi}^J + V), \tag{5}$$

we get

$$\dot{H} \equiv \frac{dH}{dt} = -\frac{1}{2M_p^2} G_{IJ} \dot{\varphi}^I \dot{\varphi}^J. \tag{6}$$

Now, let's consider the perturbed scalar fields as $\varphi^I(t) + \delta\varphi^I(t, \mathbf{x})$, and define the curvature perturbation. Here the curvature perturbation refers

to the uniform density curvature perturbation, ζ, which is still equivalent to the comoving curvature perturbation on super horizon scales in multiple field inflationary models. The curvature perturbation is defined as the difference between an initial space-flat fixed-t slice and a final uniform energy density fixed-t slice (see [20, 11, 15] for details),

$$\zeta(t,\mathbf{x}) = \delta N = H\delta t, \tag{7}$$

where $N = \int H dt$ is the integrated number of e-folds. Following the argument in [11], we expand the curvature perturbation to the second order,

$$\zeta \simeq N_{,I}(t)\delta\varphi^I(\mathbf{x}) + \frac{1}{2}N_{,IJ}(t)\delta\varphi^I(\mathbf{x})\delta\varphi^J(\mathbf{x}), \tag{8}$$

where $N_{,I} = \frac{\partial N}{\partial \varphi^I}$, $N_{,IJ} = \frac{\partial^2 N}{\partial\varphi^I\partial\varphi^J}$. In this equation, it is the partial differentiation, not the covariant differentiation that is used, which is the same as in [8]. This is due to the definition of the curvature perturbation.

3 the non-linear parameter, f_{NL}

The non Gaussianity of the curvature is expressed in the form

$$\zeta = \zeta_g - \frac{3}{5}f_{NL}(\zeta_g^2 - \langle\zeta_g^2\rangle), \tag{9}$$

where ζ_g is Gaussian, with $\langle\zeta_g\rangle = 0$, and $f_{NL} is the non-linear parameter$. For a generic cosmological perturbation, $\psi(t,\mathbf{x})$, we define its Fourier components as $\psi(\mathbf{k}) = \int d^3x\psi(t,\mathbf{x})e^{i\mathbf{k}\cdot\mathbf{x}}$. Then, using Eq.(9), we get

$$\zeta(\mathbf{k}) = \zeta_g(\mathbf{k}) - \frac{3}{5}f_{NL}\{\int\frac{d^3k_1}{(2\pi)^3}[\zeta_g(\mathbf{k}_1)\zeta_g(\mathbf{k}-\mathbf{k}_1)] - (2\pi)^3\delta^3(\mathbf{k})\langle\zeta_g^2\rangle\} \tag{10}$$

On the other hand, using Eq.(8), we get

$$\zeta(\mathbf{k}) = N_{,I}\delta\varphi^I(\mathbf{k}) + \frac{1}{2}N_{,IJ}\{\int\frac{d^3k_1}{(2\pi)^3}[\delta\varphi^I(\mathbf{k}_1)\delta\varphi^J(\mathbf{k}-\mathbf{k}_1)] - (2\pi)^3\delta^3(\mathbf{k})\langle\delta\varphi^I\delta\varphi^J\rangle\}. \tag{11}$$

Here, in order to keep that $\langle\zeta\rangle = 0$, we have added the term, $-N_{IJ}\delta\varphi^I\delta\varphi^J$, to the left-hand side of Eq.(8). And we have supposed that $\langle\zeta_g^2\rangle$ and $\langle\delta\varphi^I\delta\varphi^J\rangle$ are independent of the spatial coordinates.

The spectrum of ζ_g, $P_\zeta(k)$, is defined in the standard way by

$$\langle \zeta_g(\mathbf{k}_1)\zeta_g(\mathbf{k}_2)\rangle = (2\pi)^3\delta^3(\mathbf{k}_1+\mathbf{k}_2)P_\zeta(k_1), \tag{12}$$

with $k \equiv |\mathbf{k}|$. Together with Eq.(10), to first order of f_{NL}, the spectrum of ζ is

$$\langle \zeta(\mathbf{k}_1)\zeta(\mathbf{k}_2)\rangle \simeq \langle \zeta_g(\mathbf{k}_1)\zeta_g(\mathbf{k}_2)\rangle = (2\pi)^3\delta^3(\mathbf{k}_1+\mathbf{k}_2)P_\zeta(k_1). \tag{13}$$

For this multiple field model, by assuming the quasi exponential inflation, basing on the results in [15, 16], we may define the spectrum of the scalar fields as

$$\langle \delta\varphi^I(\mathbf{k}_1)\delta\varphi^J(\mathbf{k}_2)\rangle = (2\pi)^3\delta^3(\mathbf{k}_1+\mathbf{k}_2)P_{\delta\varphi}(k_1)G^{IJ}(\varphi_*), \tag{14}$$

with $\frac{k^3}{2\pi^2}P_{\delta\varphi}(k) = (\frac{H_*}{2\pi})^2$. The subscript, $*$, means the value calculated at the moment that the corresponding scale crosses out the Hubble horizon, $k = aH$. Using the equations (11) and (14), to leading order, we get the other expression of the spectrum of ζ,

$$\langle \zeta(\mathbf{k}_1)\zeta(\mathbf{k}_2)\rangle \simeq (2\pi)^3\delta^3(\mathbf{k}_1+\mathbf{k}_2)P_{\delta\varphi}(k_1)N_{,I}N_{,J}G^{IJ}. \tag{15}$$

Comparing Eq.(13) with Eq.(15), we obtain the relation between $P_\zeta(k)$ and $P_{\delta\varphi}(k)$,

$$P_\zeta(k) = P_{\delta\varphi}(k)N_{,I}N_{,J}G^{IJ}. \tag{16}$$

Now let's calculate the bispectrum of ζ. Using Eq.(10), to the first order of f_{NL}, we get

$$\langle \zeta(\mathbf{k}_1)\zeta(\mathbf{k}_2)\zeta(\mathbf{k}_3)\rangle = (2\pi)^3\delta^3(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3) \times \{-\frac{6}{5}[P_\zeta(k_1)P_\zeta(k_2) + cyclic]\}, \tag{17}$$

where *cyclic* refers to the term, $P_\zeta(k_2)P_\zeta(k_3)+P_\zeta(k_3)P_\zeta(k_1)$. Above, we have used Eq.(12). On the other side, using Eq.(11), we get

$$\langle \zeta(\mathbf{k}_1)\zeta(\mathbf{k}_2)\zeta(\mathbf{k}_3)\rangle \simeq (2\pi)^3\delta^3(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)B(k_1,k_2,k_3), \tag{18}$$

with

$$B(k_1,k_2,k_3) \equiv N_{,I}N_{,J}N_{,KL}G^{IK}G^{JL}[P_{\delta\varphi}(k_1)P_{\delta\varphi}(k_2) + cyclic], \tag{19}$$

where *cyclic* refers to the term, $P_{\delta\varphi}(k_2)P_{\delta\varphi}(k_3) + P_{\delta\varphi}(k_3)P_{\delta\varphi}(k_1)$. Here, we note that, in Eq.(18), we have ignored the contribution of the term,

$$N_{,I}N_{,J}N_{,K}\langle\delta\varphi^I(\mathbf{k}_1)\delta\varphi^J(\mathbf{k}_2)\delta\varphi^K(\mathbf{k}_3)\rangle, \tag{20}$$

which comes from the intrinsic non Gaussianity of $\delta\varphi^I(\mathbf{k})$. We know, that for the case with the trivial target space metric, $G_{IJ} = \delta_{IJ}$, in [17], it has been proved that the contribution of the intrinsic non Gaussianity is small enough to be neglected. In this paper, we suppose that, for nearly Gaussian perturbations, $\delta\varphi^I$, the intrinsic non Gaussianity (20) is still small enough to be neglected in the context of slow-roll inflation, and the bispectrum of ζ can be obtained from Eq.(18).

Comparing Eq.(17) and Eq.(18), we get the non-linear parameter as

$$f_{NL} = -\frac{5}{6} \times \frac{G^{IM}G^{KN}N_{,I}N_{,K}N_{,MN}}{(N_{,I}N_{,J}G^{IJ})^2}. \tag{21}$$

This is an important result of this paper. Although this expression is the same as the second term on the right-hand side of Eq.(38) in Ref.[8], here we obtain it for general multi-field inflationary models.

4 f_{NL} and slow-rolling parameters

In this section, with the slow-rolling condition, we try to express f_{NL} in term of the slow-rolling parameters. So we define some parameters. The first is ε defined as

$$\varepsilon = -\frac{\dot{H}}{H^2}. \tag{22}$$

Using Eq.(6), we get

$$\varepsilon = \frac{G_{IJ}\dot{\varphi}^I\dot{\varphi}^J}{2M_p^2H^2}. \tag{23}$$

Now let's use the slow-rolling approximation. Then Eq.(3) becomes

$$3H\dot{\varphi}^I + G^{IJ}V_{,J} \simeq 0 \Rightarrow \dot{\varphi}^I \simeq -\frac{G^{IJ}V_{,J}}{3H} \tag{24}$$

And Eq.(5) becomes

$$H^2 \simeq \frac{1}{3M_p^2}V. \tag{25}$$

So ε can rewritten approximately as

$$\varepsilon \simeq \frac{G^{IJ}V_{,I}V_{,J}M_p^2}{2V^2}. \tag{26}$$

Then we define another parameter, ε_I, as

$$\varepsilon_I \equiv -\frac{V_{,I}M_p}{\sqrt{2}V}. \tag{27}$$

This implies an relation, $\varepsilon = G_{IJ}\varepsilon^I\varepsilon^J$.

In order to express f_{NL} by the slow-rolling parameters, we should firstly get the expression of $N_{,I}$. From Eq.(7), we get

$$\delta N = H\delta t = -\frac{1}{\varepsilon}d\ln H \simeq -\frac{1}{\varepsilon}d\ln\sqrt{V} = -\frac{1}{2\varepsilon V}V_{,I}\delta\varphi^I. \tag{28}$$

Then it may be supposed that we can extract the derivation of N with respect to φ^I,

$$N_{,I} \simeq -\frac{1}{2\varepsilon V}V_{,I} = \frac{\varepsilon_I}{\sqrt{2}\varepsilon M_p}. \tag{29}$$

However, this equation can not be applied to Eq.(21) unless the perturbations during multi-field inflation are purely adiabatic. In fact, for a general multi-field model, the entropy perturbation do exist. (See Ref.[21] for an extensive explanation.)

In order to use Eq.(29), in this section, we impose the condition: *Adiabatic Perturbations*. This implies that the result in this section is only applicable to the multi-field inflation during which the perturbations are purely adiabatic.

Then we get

$$G^{IJ}N_{,I}N_{,J} = \frac{1}{2\varepsilon M_p^2}. \tag{30}$$

The derivation of ε_I or ε with respect to φ^J can be expressed approximately as

$$\begin{aligned} \frac{\partial\varepsilon_I}{\partial\varphi^J} &\simeq -\frac{\partial}{\partial\varphi^J}\left(\frac{V_{,I}M_p}{\sqrt{2}V}\right) = \frac{\sqrt{2}}{M_p}(\varepsilon_I\varepsilon_J - \frac{1}{2}\eta_{IJ}), && (31)\\ \frac{\partial\varepsilon}{\partial\varphi^J} &= \frac{\partial}{\partial\varphi^J}(G^{IJ}\varepsilon_I\varepsilon_J) \simeq G^{KL}{}_{,J}\varepsilon_K\varepsilon_L + \frac{\sqrt{2}}{M_p}(2\varepsilon\varepsilon_J - G^{KL}\varepsilon_K\eta_{LJ}), && (32) \end{aligned}$$

with $\eta_{IJ} \equiv \frac{V_{,IJ}M_p^2}{V}$ and $G^{KL}{}_{,J} \equiv \frac{\partial G^{KL}}{\partial \varphi^J}$. Now It is the time to calculate $N_{,IJ}$,

$$
\begin{aligned}
N_{,IJ} &= \frac{\partial}{\partial \varphi^J}\left(\frac{\varepsilon_I}{\sqrt{2}\varepsilon}\right) \\
&\simeq \frac{1}{\varepsilon^2 M_p^2}\{G^{KL}\varepsilon_K \eta_{LJ}\varepsilon_I - \frac{G^{KL}{}_{,J}}{\sqrt{2}M_p}\varepsilon_K\varepsilon_L\varepsilon_I - \varepsilon\varepsilon_I\varepsilon_J - \frac{1}{2}\varepsilon\eta_{IJ}\}. \qquad (33)
\end{aligned}
$$

Now it is easy to get

$$
G^{IK}G^{JL}N_{,I}N_{,J}N_{,KL} = -\frac{1}{2\varepsilon^3 M_p^3}\beta, \qquad (34)
$$

with

$$
\beta \equiv \frac{\varepsilon^2}{M_p} + \frac{G^{MN}{}_{,L}}{\sqrt{2}}G^{JL}\varepsilon_M\varepsilon_N\varepsilon_J - \frac{1}{2M_p}G^{JL}G^{MN}\varepsilon_J\varepsilon_M\eta_{NL}. \qquad (35)
$$

Substituting Eq.(30) and (34) into Eq.(21), we can express the nonlinear parameter in the form as

$$
f_{NL} \simeq \frac{5}{3} \times \frac{\beta M_p}{\varepsilon} \qquad (36)
$$

Here we emphasize again that Eq.(36) is only applicable to the multi-field models in which the entropy perturbations can be neglected.

5 Summary

In this paper, the "δN-formalism" suggested to express the non Gaussianity in [11] is generalized to the multi-field inflationary models with the non-trivial target space metric. One key step in our derivation is the equation (14). We believe that this equation is correct [15, 16]. We have rewritten the result by using the slow-rolling approximation, which is easy to be analyzed. But this result is obtained under the condition of *Adiabatic Perturbations*. For a general multi-field model, we should use Eq.(21) to calculate f_{NL}.

Additionally, in this paper we have restricted our attention to the contribution of the Gaussian part of $\delta\varphi^I$ and ignored the contribution of the intrinsic non Gaussianity of $\delta\varphi^I$, (20). We emphasize that in some case the term (20) should be included. This will be discussed in future work.

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